Monodromy at infinity of A-hypergeometric functions and toric compactifications *

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Abstract

We study A-hypergeometric functions introduced by Gelfand-Kapranov-Zelevinsky [4] and prove a formula for the eigenvalues of their monodromy automorphisms defined by the analytic continuaions along large loops contained in complex lines parallel to the coordinate axes. The method of toric compactifications introduced in [12] and [16] will be used to prove our main theorem.

1 Introduction

The theory of A-hypergeometric systems introduced by Gelfand-Kapranov-Zelevinsky [4] is an ultimate generalization of that of classical hypergeometric differential equations. As in the case of hypergeometric equations, the holomorphic solutions to A-hypergeometric systems admit power series expansions [4] and integral representations [5]. Moreover this theory has very deep connections with other fields of mathematics, such as toric varieties, projective duality, period integrals, mirror symmetry and combinatorics. Also from the viewpoint of $\mathcal{D}$-module theory ([2], [8] and [9] etc.), A-hypergeometric $\mathcal{D}$-modules are very elegantly constructed in [5]. For the recent development of this subject see [20] etc. However, to the best of our knowledge, it seems that the monodromy representations of their solutions i.e. A-hypergeometric functions are not completely studied yet. One of the most successful attempts toward the understanding of these monodromy representations would be Borisov-Horja's Mellin-Barnes type connection formulas for A-hypergeometric functions in [1] and [7]. In this paper, we study the monodromy representations in the light of the theory of $\mathcal{D}$-modules and constructible sheaves. In particular, we give a formula for the characteristic polynomials of the monodromy automorphisms of A-hypergeometric functions obtained by the analytic continuations along large loops contained in complex lines parallel to the coordinate axes. Namely we study the monodromy at infinity of A-hypergeometric functions (as for the topological monodromy at infinity in the theory of polynomial maps, see [12] and [16] etc.). Our result are general and will be described only by the configulation A and the parameter $\gamma \in \mathbb{C}^n$ which are used to define the A-hypergeometric system.

In order to give the precise statement of our theorem, let us recall the definition of A-hypergeometric systems in [4] and [5]. Let $A = \{a(1), a(2), \ldots, a(m)\} \subset \mathbb{Z}^{n-1}$ be a

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finite subset of the lattice $\mathbb{Z}^{n-1}$. Assume that A generate $\mathbb{Z}^{n-1}$ as an affine lattice. Then the convex hull Q of A in $\mathbb{R}^{n-1}_v = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}^{n-1}$ is an $(n-1)$-dimensional polytope. For $j = 1, 2, \dots, m$ set $\widetilde{a(j)} := (a(j), 1) \in \mathbb{Z}^n = \mathbb{Z}^{n-1} \oplus \mathbb{Z}$ and consider the $n \times m$ integer matrix

$$\widetilde{A} := \begin{pmatrix} {}^t\widetilde{a(1)} & {}^t\widetilde{a(2)} & \cdots & {}^t\widetilde{a(m)} \end{pmatrix} = (a_{ij}) \in M(n, m, \mathbb{Z}) \tag{1.1}$$

whose j-th column is ${}^t\widetilde{a(j)}$. Then the GKZ hypergeometric system on $X = \mathbb{C}^A = \mathbb{C}^m_z$ associated with $A \subset \mathbb{Z}^{n-1}$ and a parameter $\gamma \in \mathbb{C}^n$ is

$$\left(\sum_{j=1}^m a_{ij} z_j \frac{\partial}{\partial z_j} - \gamma_i \right) f(z) = 0 \quad (1 \leq i \leq n), \tag{1.2}$$

$$\left\{ \prod_{\mu_j > 0} \left(\frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left(\frac{\partial}{\partial z_j} \right)^{-\mu_j} \right\} f(z) = 0 \quad (\mu \in \mathrm{Ker} \widetilde{A} \cap \mathbb{Z}^m \setminus \{0\}) \tag{1.3}$$

(see [4] and [5]). Let $\mathcal{D}_X$ be the sheaf of differential operators with holomorphic coefficients on $X = \mathbb{C}^m_z$ and set

$$P_i := \sum_{j=1}^m a_{ij} z_j \frac{\partial}{\partial z_j} - \gamma_i \quad (1 \leq i \leq n), \tag{1.4}$$

$$\Box_\mu := \prod_{\mu_j > 0} \left(\frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left(\frac{\partial}{\partial z_j} \right)^{-\mu_j} \quad (\mu \in \mathrm{Ker} \widetilde{A} \cap \mathbb{Z}^m \setminus \{0\}). \tag{1.5}$$

Then the coherent $\mathcal{D}_X$-module

$$\mathcal{M}_{A,\gamma} = \mathcal{D}_X / \Big(\sum_{1 \leq i \leq n} \mathcal{D}_X P_i + \sum_{\mu \in \mathrm{Ker} \widetilde{A} \cap \mathbb{Z}^m \setminus \{0\}} \mathcal{D}_X \Box_\mu \Big) \tag{1.6}$$

which corresponds to the above system is holonomic and its solution complex

$$\mathrm{Sol}(\mathcal{M}_{A,\gamma}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_{A,\gamma}, \mathcal{O}_X) \tag{1.7}$$

is a local system on an open dense subset of X. It is well-known after [4] and [5] that the singular locus of $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$ is described by the A-discriminant varieties studied precisely by [6] (see also [14] etc.).

Let us fix an integer $j_0 \in \mathbb{Z}$ such that $1 \leq j_0 \leq m$ and for $(c_1, c_2, \dots, c_{j_0-1}, c_{j_0+1}, \dots, c_m) \in \mathbb{C}^{m-1}$ consider a complex line $L \simeq \mathbb{C}$ in $X = \mathbb{C}^m_z$ defined by

$$L = \{ z = (z_1, z_2, \dots, z_m) \in \mathbb{C}^m \mid z_j = c_j \text{ for } j \neq j_0 \} \subset X = \mathbb{C}^m_z. \tag{1.8}$$

Since L is a line parallel to the j_0-th axis $\mathbb{C}_{z_{j_0}}$ of $X = \mathbb{C}^m_z$, we naturally identify it with $\mathbb{C}_{z_{j_0}}$. For simplicity, we denote the j_0-th coordinate function $z_{j_0} : X = \mathbb{C}^m_z \longrightarrow \mathbb{C}$ by s. Then it is well-known that if $(c_1, c_2, \dots, c_{j_0-1}, c_{j_0+1}, \dots, c_m) \in \mathbb{C}^{m-1}$ is generic there exists a finite subset $S_L \subset L \simeq \mathbb{C}_s$ such that $\mathrm{Sol}(\mathcal{M}_{A,\gamma})|_L$ is a local system on $L \setminus S_L$. Let us take such a line L in $X = \mathbb{C}^m_z$ and a point $s_0 \in L \simeq \mathbb{C}_s$ in L such that $|s_0| > \max_{s \in S_L} |s|$. Then we obtain a monodromy automorphism

$$\mathrm{Sol}(\mathcal{M}_{A,\gamma})_{s_0} \xrightarrow{\sim} \mathrm{Sol}(\mathcal{M}_{A,\gamma})_{s_0} \tag{1.9}$$

defined by the analytic continuation of the sections of $\mathrm{Sol}(\mathcal{M}_{A,\gamma})|_L$ along the path

$$C_{s_0} = \{s_0 \exp(\sqrt{-1}\theta) \mid 0 \leq \theta \leq 2\pi\} \tag{1.10}$$

in $L \simeq \mathbb{C}_s$. Since the characteristic polynomial of this automorphism does not depend on L and $s_0 \in L$, we denote it simply by $\lambda_{j_0}^{\infty}(t) \in \mathbb{C}[t]$. We call $\lambda_{j_0}^{\infty}(t)$ the characteristic polynomial of the j_0-th monodromy at infinity of the A-hypergeometric functions $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$. According to the fundamental result of [4], if $\gamma \in \mathbb{C}^n$ is generic (i.e. non-resonant in the sense of [5, Section 2.9]), the rank of the local system $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$ is equal to the normalized $(n-1)$-dimensional volume $\mathrm{Vol}_{\mathbb{Z}}(Q) \in \mathbb{Z}$ of Q with respect to the lattice $\mathbb{Z}^{n-1}$. See also Saito-Sturmfels-Takayama [19] for the related results. Therefore, the degree of the the characteristic polynomial $\lambda_{j_0}^{\infty}(t)$ is $\mathrm{Vol}_{\mathbb{Z}}(Q)$. In order to give a formula for $\lambda_{j_0}^{\infty}(t) \in \mathbb{C}[t]$ we shall prepare some notations. First, we set $\alpha = \gamma_n, \beta_1 = -\gamma_1 - 1, \beta_2 = -\gamma_2 - 1, \dots, \beta_{n-1} = -\gamma_{n-1} - 1$ and $\beta = (\beta_1, \beta_2, \dots, \beta_{n-1}) \in \mathbb{C}^{n-1}$ (see [5, Theorem 2.7]). Next let $\Delta_1, \Delta_2, \dots, \Delta_k$ be the $(n-2)$-dimensional faces i.e. the facets of Q such that $a(j_0) \notin \Delta_r$ $(r = 1, 2, \dots, k)$. Then for each $r = 1, 2, \dots, k$ there exists a unique primitive vector $u^r \in \mathbb{Z}^{n-1} \setminus \{0\}$ such that

$$\Delta_r = \{v \in Q \mid \langle u^r, v\rangle = \min_{w \in Q} \langle u^r, w\rangle\}. \tag{1.11}$$

Let us set

$$h_r = \min_{w \in Q} \langle u^r, w\rangle = \langle u^r, \Delta_r\rangle \in \mathbb{Z}, \tag{1.12}$$

$$d_r = \langle u^r, a(j_0)\rangle - h_r \in \mathbb{Z}. \tag{1.13}$$

Since $-u^r \in \mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}$ is the primitive outer conormal vector of the facet $\Delta_r \prec Q$ of Q and we have

$$d_r = \langle -u^r, w - a(j_0)\rangle \tag{1.14}$$

for any $w \in \Delta_r$, the integer d_r is the lattice distance of the point $a(j_0) \in Q$ from Δ_r. In particular, we have $d_r > 0$. Finally we set

$$\delta_r = \alpha h_r + \langle \beta, u^r\rangle \in \mathbb{C} \tag{1.15}$$

for $r = 1, 2, \dots, k$. Then we obtain the following theorem.

Theorem 1.1. *Assume that $\gamma \in \mathbb{C}^n$ is non-resonant in the sense of [5, Section 2.9]. Then the characteristic polynomial $\lambda_{j_0}^{\infty}(t)$ of the j_0-th monodromy at infinity of* $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$ *is given by*

$$\lambda_{j_0}^{\infty}(t) = \prod_{r=1}^{k} \left\{ t^{d_r} - \exp(-2\pi\sqrt{-1}\delta_r) \right\}^{\mathrm{Vol}_{\mathbb{Z}}(\Delta_r)}, \tag{1.16}$$

where $\mathrm{Vol}_{\mathbb{Z}}(\Delta_r) \in \mathbb{Z}$ *is the normalized $(n-2)$-dimensional volume of Δ_r.*

For the proof of this theorem, we will use some sheaf-theoretical methods such as nearby and constructible sheaves, and a toric compactification of the algebraic torus $T = (\mathbb{C}^*)^n \subset (\mathbb{C}^*)^{n-1} \times L$ similar to the one used in the study of topological monodromy at infinity of polynomial maps in [12] and [16]. As well as Bernstein-Khovanskii-Kushnirenko's theorem [10], the general results on nearby cycle sheaves of local systems obtained in [15, Section 5] will play an important role in the proof.

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2 Preliminary notions and results

In this section, we introduce basic notions and results which will be used in the proof of our main theorem. We essentially follow the terminology of [2], [8] and [9]. For example, for a topological space X we denote by $\mathbf{D}^b(X)$ the derived category whose objects are bounded complexes of sheaves of $\mathbb{C}_X$-modules on X.

Definition 2.1. *Let X be an algebraic variety over $\mathbb{C}$. Then*

1. *We say that a sheaf $\mathcal{F}$ on X is constructible if there exists a stratification $X = \bigsqcup_\alpha X_\alpha$ of X such that $\mathcal{F}|_{X_\alpha}$ is a locally constant sheaf of finite rank for any α.*

2. *We say that an object $\mathcal{F}$ of $\mathbf{D}^b(X)$ is constructible if the cohomology sheaf $H^j(\mathcal{F})$ of $\mathcal{F}$ is constructible for any $j \in \mathbb{Z}$. We denote by $\mathbf{D}^b_c(X)$ the full subcategory of $\mathbf{D}^b(X)$ consisting of constructible objects $\mathcal{F}$.*

Recall that for any morphism $f\colon X \longrightarrow Y$ of algebraic varieties over $\mathbb{C}$ there exists a functor

$$Rf_*\colon \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y) \tag{2.1}$$

of direct images. This functor preserves the constructibility and we obtain also a functor

$$Rf_*\colon \mathbf{D}^b_c(X) \longrightarrow \mathbf{D}^b_c(Y). \tag{2.2}$$

For other basic operations $Rf_!$, f^{-1}, $f^!$ etc. in derived categories, see [9] for the detail.

Next we introduce the notion of constructible functions.

Definition 2.2. *Let X be an algebraic variety over $\mathbb{C}$ and G an abelian group. Then we say a G-valued function $\rho\colon X \longrightarrow G$ on X is constructible if there exists a stratification $X = \bigsqcup_\alpha X_\alpha$ of X such that $\rho|_{X_\alpha}$ is constant for any α. We denote by $\mathrm{CF}_G(X)$ the abelian group of G-valued constructible functions on X.*

Let $\mathbb{C}(t)^* = \mathbb{C}(t) \setminus \{0\}$ be the multiplicative group of the function field $\mathbb{C}(t)$ of the scheme $\mathbb{C}$. In this paper, we consider $\mathrm{CF}_G(X)$ only for $G = \mathbb{Z}$ or $\mathbb{C}(t)^*$. For a G-valued constructible function $\rho\colon X \longrightarrow G$, by taking a stratification $X = \bigsqcup_\alpha X_\alpha$ of X such that $\rho|_{X_\alpha}$ is constant for any α as above, we set

$$\int_X \rho := \sum_\alpha \chi(X_\alpha) \cdot \rho(x_\alpha) \in G, \tag{2.3}$$

where x_α is a reference point in X_α. Then we can easily show that $\int_X \rho \in G$ does not depend on the choice of the stratification $X = \bigsqcup_\alpha X_\alpha$ of X. Hence we obtain a homomorphism

$$\int_X \colon \mathrm{CF}_G(X) \longrightarrow G \tag{2.4}$$

of abelian groups. For $\rho \in \mathrm{CF}_G(X)$, we call $\int_X \rho \in G$ the topological (Euler) integral of ρ over X. More generally, for any morphism $f\colon X \longrightarrow Y$ of algebraic varieties over $\mathbb{C}$ and $\rho \in \mathrm{CF}_G(X)$, we define the push-forward $\int_f \rho \in \mathrm{CF}_G(Y)$ of ρ by

$$\left(\int_f \rho\right)(y) := \int_{f^{-1}(y)} \rho \tag{2.5}$$

for $y \in Y$. This defines a homomorphism

$$\int_f : \mathrm{CF}_G(X) \longrightarrow \mathrm{CF}_G(Y) \tag{2.6}$$

of abelian groups.

Among various operations in derived categories, the following nearby and vanishing cycle functors introduced by Deligne will be frequently used in this paper (see [2, Section 4.2] for an excellent survey of this subject). Let $f : X \longrightarrow \mathbb{C}$ be a regular function on an algebraic variety X over $\mathbb{C}$ and set $X_0 = \{x \in X \mid f(x) = 0\}$. Then there exist functors

$$\psi_f,\ \varphi_f : \mathbf{D}^b_c(X) \longrightarrow \mathbf{D}^b_c(X_0). \tag{2.7}$$

which are called the nearby and vanishing cycle functors of f respectively. As we see in the next proposition, the nearby cycle functor ψ_f generalizes the classical notion of Milnor fibers. First, let us recall the definition of Milnor fibers and Milnor monodromies over singular varieties (see for example [22] for a review on this subject). Let X be a subvariety of $\mathbb{C}^m$ and $f : X \longrightarrow \mathbb{C}$ a non-constant regular function on X. Namely we assume that there exists a polynomial function $\widetilde{f} : \mathbb{C}^m \longrightarrow \mathbb{C}$ on $\mathbb{C}^m$ such that $\widetilde{f}|_X = f$. For simplicity, assume also that the origin $0 \in \mathbb{C}^m$ is contained in $X_0 = \{x \in X \mid f(x) = 0\}$. Then the following lemma is well-known (see for example [17] and [13, Definition 1.4]).

Lemma 2.3. *For sufficiently small $\varepsilon > 0$, there exists $\eta_0 > 0$ with $0 < \eta_0 \ll \varepsilon$ such that for $0 < \forall \eta < \eta_0$ the restriction of f:*

$$X \cap B(0;\varepsilon) \cap \widetilde{f}^{-1}(D(0;\eta) \setminus \{0\}) \longrightarrow D(0;\eta) \setminus \{0\} \tag{2.8}$$

is a topological fiber bundle over the punctured disk $D(0;\eta) \setminus \{0\} = \{z \in \mathbb{C} \mid 0 < |z| < \eta\}$, where $B(0;\varepsilon)$ is the open ball in $\mathbb{C}^m$ with radius ε centered at the origin.

Definition 2.4. *A fiber of the above fibration is called the Milnor fiber of $f : X \longrightarrow \mathbb{C}$ at $0 \in X$ and we denote it by F_0.*

Similarly, for $x \in X_0$ we define the Milnor fiber F_x of f at x.

Proposition 2.5. ([2, Proposition 4.2.2]) *For any $x \in X_0$ and $j \in \mathbb{Z}$ there exists a natural isomorphism*

$$H^j(F_x;\mathbb{C}) \simeq H^j(\psi_f(\mathbb{C}_X))_x. \tag{2.9}$$

By this proposition, we can study the cohomology groups $H^j(F_x;\mathbb{C})$ of the Milnor fiber F_x by using sheaf theory. Recall also that in the above situation, as in the case of polynomial functions over $\mathbb{C}^n$ (see [17]), we can define the Milnor monodromy operators

$$\Phi_{j,x} : H^j(F_x;\mathbb{C}) \xrightarrow{\sim} H^j(F_x;\mathbb{C}) \qquad (j = 0, 1, \ldots) \tag{2.10}$$

and the zeta-function

$$\zeta_{f,x}(t) := \prod_{j=0}^{\infty} \det\left(\mathrm{id} - t\Phi_{j,x}\right)^{(-1)^j} \tag{2.11}$$

associated with it. Since the above product is in fact finite, $\zeta_{f,x}(t)$ is a rational function of t and its degree in t is the topological Euler characteristic $\chi(F_x)$ of the Milnor fiber F_x. For the explicit formulas of $\zeta_{f,x}(t)$ and $\chi(F_x)$, see [11], [17], [23] and [15] etc. This classical notion of Milnor monodromy zeta functions can be also generalized as follows.

Definition 2.6. *Let $f\colon X \longrightarrow \mathbb{C}$ be a non-constant regular function on X and $\mathcal{F} \in \mathbf{D}^b_c(X)$. Set $X_0 := \{x \in X \mid f(x) = 0\}$. Then there exists a monodromy automorphism*

$$\Phi(\mathcal{F})\colon \psi_f(\mathcal{F}) \xrightarrow{\sim} \psi_f(\mathcal{F}) \tag{2.12}$$

of $\psi_f(\mathcal{F})$ in $\mathbf{D}^b_c(X_0)$ (see [2, Section 4.2]). We define a $\mathbb{C}(t)^*$-valued constructible function $\zeta_f(\mathcal{F}) \in \mathrm{CF}_{\mathbb{C}(t)^*}(X_0)$ on X_0 by

$$\zeta_{f,x}(\mathcal{F})(t) := \prod_{j\in\mathbb{Z}} \det\left\{\mathrm{id} - t\Phi(\mathcal{F})_{j,x}\right\}^{(-1)^j} \tag{2.13}$$

for $x \in X_0$, where $\Phi(\mathcal{F})_{j,x}\colon (H^j(\psi_f(\mathcal{F})))_x \xrightarrow{\sim} (H^j(\psi_f(\mathcal{F})))_x$ is the stalk at $x \in X_0$ of the sheaf homomorphism

$$\Phi(\mathcal{F})_j\colon H^j(\psi_f(\mathcal{F})) \xrightarrow{\sim} H^j(\psi_f(\mathcal{F})) \tag{2.14}$$

induced by $\Phi(\mathcal{F})$.

The following proposition will play an important role in the proof of our main theorem. For the proof, see for example, [2, p.170-173] and [21].

Proposition 2.7. *Let $\pi\colon Y \longrightarrow X$ be a proper morphism of algebraic varieties over $\mathbb{C}$ and $f\colon X \longrightarrow \mathbb{C}$ a non-constant regular function on X. Set $g := f \circ \pi\colon Y \longrightarrow \mathbb{C}$, $X_0 := \{x \in X \mid f(x) = 0\}$ and $Y_0 := \{y \in Y \mid g(y) = 0\} = \pi^{-1}(X_0)$. Then for any $\mathcal{G} \in \mathbf{D}^b_c(Y)$ we have*

$$\int_{\pi|_{Y_0}} \zeta_g(\mathcal{G}) = \zeta_f(R\pi_*\mathcal{G}) \tag{2.15}$$

in $\mathrm{CF}_{\mathbb{C}(t)^*}(X_0)$, where

$$\int_{\pi|_{Y_0}} \colon \mathrm{CF}_{\mathbb{C}(t)^*}(Y_0) \longrightarrow \mathrm{CF}_{\mathbb{C}(t)^*}(X_0) \tag{2.16}$$

is the push-forward of $\mathbb{C}(t)^*$-valued constructible functions by $\pi|_{Y_0}\colon Y_0 \longrightarrow X_0$.

For the proof of the following propositions, see [15, Proposition 5.2 and 5.3].

Proposition 2.8. *([15, Proposition 5.2 (iii)]) Let $\mathcal{L}$ be a local system of rank $r > 0$ on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Denote by $A \in GL_r(\mathbb{C})$ the monodromy matrix of $\mathcal{L}$ along the loop $\{e^{i\theta} \mid 0 \leq \theta \leq 2\pi\}$ in $\mathbb{C}^*$, which is defined up to conjugacy. Let $j : \mathbb{C}^* \hookrightarrow \mathbb{C}$ be the inclusion and h a function on $\mathbb{C}$ defined by $h(z) = z^m$ ($m \in \mathbb{Z}_{>0}$) for $z \in \mathbb{C}$. Then we have*

$$\zeta_{h,0}(j_!\mathcal{L})(t) = \det(\mathrm{id} - t^m A) \in \mathbb{C}(t)^*. \tag{2.17}$$

Proposition 2.9. *([15, Proposition 5.3]) Let $\mathcal{L}$ be a local system on $(\mathbb{C}^*)^k$ for $k \geq 2$ and $j\colon (\mathbb{C}^*)^k \hookrightarrow \mathbb{C}^k$ the inclusion. Let $h : \mathbb{C}^k \longrightarrow \mathbb{C}$ be a function defined by $h(z) = z_1^{m_1} z_2^{m_2} \cdots z_k^{m_k} \not\equiv 1$ ($m_i \in \mathbb{Z}_{\geq 0}$) for $z \in \mathbb{C}^k$. Then the monodromy zeta function $\zeta_{h,0}(j_!\mathcal{L})(t)$ of $j_!\mathcal{L} \in \mathbf{D}^b_c(\mathbb{C}^k)$ at $0 \in \mathbb{C}^k$ is $1 \in \mathbb{C}(t)^*$.*

3 Monodromy at infinity of A-hypergeometric functions

In this section, we inherit the notations and the situation in the introduction and prove our main theorem (Theorem 1.1). First, let us recall the definition of the non-resonance of the parameter $\gamma \in \mathbb{C}^n$ introduced in [5, Section 2.9]. Let K be a convex cone in $\mathbb{R}^n$ generated by the vectors $(a(1),1),(a(2),1),\dots,(a(m),1) \in \mathbb{Z}^n = \mathbb{Z}^{n-1} \oplus \mathbb{Z}$. For each face $\Gamma \prec K$ of K denote by $\mathrm{Lin}(\Gamma) \simeq \mathbb{C}^{\dim \Gamma}$ the $\mathbb{C}$-linear span of Γ in $\mathbb{C}^n$.

Definition 3.1. *([5, Section 2.9]) We say that the parameter $\gamma \in \mathbb{C}^n$ is non-resonant if for any face $\Gamma \prec K$ of codimension one we have $\gamma \notin \mathbb{Z}^n + \mathrm{Lin}(\Gamma)$.*

By the fundamental result of [4], if $\gamma \in \mathbb{C}^n$ is non-resonant the generic rank of $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$ is equal to the normalized $(n-1)$-dimensional volume $\mathrm{Vol}_{\mathbb{Z}}(Q) \in \mathbb{Z}$ of Q. Therefore, for any $j_0 \in \mathbb{Z}$ such that $1 \leq j_0 \leq m$ the degree of the the characteristic polynomial $\lambda_{j_0}^{\infty}(t)$ is $\mathrm{Vol}_{\mathbb{Z}}(Q)$.

Theorem 3.2. *Assume that $\gamma \in \mathbb{C}^n$ is non-resonant. Then the characteristic polynomial $\lambda_{j_0}^{\infty}(t)$ of the j_0-th monodromy at infinity of $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$ is given by*

$$\lambda_{j_0}^{\infty}(t) = \prod_{r=1}^{k} \left\{ t^{d_r} - \exp(-2\pi\sqrt{-1}\delta_r) \right\}^{\mathrm{Vol}_{\mathbb{Z}}(\Delta_r)}, \tag{3.1}$$

where $\mathrm{Vol}_{\mathbb{Z}}(\Delta_r) \in \mathbb{Z}$ is the normalized $(n-2)$-dimensional volume of Δ_r.

Proof. Let

$$L = \{ z = (z_1, z_2, \dots, z_m) \in \mathbb{C}^m \mid z_j = c_j \text{ for } j \neq j_0 \} \subset X = \mathbb{C}^m_z. \tag{3.2}$$

($(c_1, c_2, \dots, c_{j_0-1}, c_{j_0+1}, \dots, c_m) \in \mathbb{C}^{m-1}$) be the defining equation of $L \simeq \mathbb{C}_s$ in $X = \mathbb{C}^m_z$ and define a Laurent polynomial p on $(\mathbb{C}^*)^{n-1}_x \times L \simeq (\mathbb{C}^*)^{n-1}_x \times \mathbb{C}_s$ by

$$p(x,s) = s x^{a(j_0)} + \sum_{j \neq j_0} c_j x^{a(j)}. \tag{3.3}$$

Denote by $\widetilde{P}$ the convex hull of $(a(j_0),1) \sqcup \{(a(j),0) \mid j \neq j_0\}$ in $\mathbb{R}^n_{\widetilde{v}} = \mathbb{R}^{n-1}_v \oplus \mathbb{R}$. We may assume that $c_j \neq 0$ for any $j \neq j_0$ and the Newton polytope of $p(x,s)$ is $\widetilde{P}$. However note that the dimension of the polytope $\widetilde{P}$ is not necessarily equal to n. Let U be an open subset of $(\mathbb{C}^*)^{n-1}_x \times L$ defined by $U = \{(x,s) \in (\mathbb{C}^*)^{n-1} \times L \mid p(x,s) \neq 0\}$ and $\pi = s : U \longrightarrow L \simeq \mathbb{C}$ the restriction of the second projection $(\mathbb{C}^*)^{n-1} \times L \longrightarrow L$ to U. Define a local system $\mathcal{L}$ of rank one on U by

$$\mathcal{L} = \mathbb{C}\, p(x,s)^{\alpha} x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n-1}^{\beta_{n-1}}. \tag{3.4}$$

Then by [5, page 270, line 9-10] we have an isomorphism

$$\mathrm{Sol}(\mathcal{M}_{A,\gamma})|_L \simeq R\pi_! \mathcal{L}[n-1] \tag{3.5}$$

in $\mathbf{D}^b_c(L)$. Let $j : L \simeq \mathbb{C}_s \hookrightarrow \mathbb{C}_s \sqcup \{\infty\} = \mathbb{P}^1$ be the embedding and $h(s) = \frac{1}{s}$ the holomorphic function defined on an neighborhood of ∞ in $\mathbb{P}^1$ such that $\{\infty\} = \{h = 0\}$.

Then it suffices to show that the monodromy zeta function $\zeta_{h,\infty}(j_! R\pi_! \mathcal{L}[n-1])(t) \in \mathbb{C}(t)^*$ of the constructible sheaf $j_! R\pi_! \mathcal{L}[n-1] \in \mathbf{D}^b_c(\mathbb{P}^1)$ at $\infty \in \mathbb{P}^1$ is given by

$$\zeta_{h,\infty}(j_! R\pi_! \mathcal{L}[n-1])(t) = \prod_{r=1}^{k} \{1 - \exp(2\pi\sqrt{-1}\delta_r)t^{d_r}\}^{\mathrm{Vol}_{\mathbb{Z}}(\Delta_r)}. \tag{3.6}$$

Indeed, by the isomorphism $\mathbb{C}^*_s \simeq \mathbb{C}^*_h$, $h = \frac{1}{s}$ the sufficiently large circle

$$C_{s_0} = \{s_0 \exp(\sqrt{-1}\theta) \mid 0 \leq \theta \leq 2\pi\} \subset \mathbb{C}^*_s \qquad (|s_0| >> 0) \tag{3.7}$$

of anti-clockwise direction is sent to the small one

$$\widetilde{C}_{\frac{1}{s_0}} = \{\frac{1}{s_0} \exp(-\sqrt{-1}\theta) \mid 0 \leq \theta \leq 2\pi\} \subset \mathbb{C}^*_h \tag{3.8}$$

of clockwise direction. Let $T = (\mathbb{C}^*)^{n-1}_x \times \mathbb{C}^*_s \simeq (\mathbb{C}^*)^n$ be the open dense torus in $(\mathbb{C}^*)^{n-1}_x \times L$ and $j' : U \cap T \hookrightarrow T$ the inclusion. Then for the constructible sheaf $\mathcal{F} = j'_!(\mathcal{L}|_{U\cap T})$ on T and the restriction $\pi' : T \longrightarrow L \simeq \mathbb{C}_s$ of the second projection $(\mathbb{C}^*)^{n-1}_x \times L \longrightarrow L$ to T we have

$$\zeta_{h,\infty}(j_! R\pi_! \mathcal{L}[n-1])(t) = \zeta_{h,\infty}(j_! R\pi'_! \mathcal{F}[n-1])(t). \tag{3.9}$$

From now on, we shall construct a toric compactification $\overline{T}$ of T such that the meromorphic extension of the coordinate function $s = \pi' : T = (\mathbb{C}^*)^{n-1}_x \times \mathbb{C}^*_s \longrightarrow L \simeq \mathbb{C}$ to $\overline{T}$ has no point of indeterminacy and induces a holomorphic map $g : \overline{T} \longrightarrow \mathbb{P}^1$ such that $g \circ \iota = j \circ \pi'$ for $\iota : T \hookrightarrow \overline{T}$. Let $\mathbb{R}^n_{\widetilde{u}}$ be the dual vector space $(\mathbb{R}^n_{\widetilde{v}})^*$ of $\mathbb{R}^n_{\widetilde{v}} = \mathbb{R}^{n-1}_v \oplus \mathbb{R}$. We denote the dual lattice $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}) \subset \mathbb{R}^n_{\widetilde{u}}$ of $\mathbb{Z}^n \subset \mathbb{R}^n_{\widetilde{v}}$ simply by $\mathbb{Z}^n$.

Definition 3.3. *For $\widetilde{u} \in \mathbb{R}^n_{\widetilde{u}}$ we define the $\widetilde{u}$-part $p^{\widetilde{u}}(x,s)$ of the Laurent polynomial $p(x,s)$ by*

$$p^{\widetilde{u}}(x,s) = \begin{cases} \displaystyle\sum_{\substack{j \neq j_0:\ (a(j),0) \\ \in \Gamma(\widetilde{P};\widetilde{u})}} c_j x^{a(j)} & \text{if } (a(j_0),1) \notin \Gamma(\widetilde{P};\widetilde{u})\ , \\ s x^{a(j_0)} + \displaystyle\sum_{\substack{j \neq j_0:\ (a(j),0) \\ \in \Gamma(\widetilde{P};\widetilde{u})}} c_j x^{a(j)} & \text{if } (a(j_0),1) \in \Gamma(\widetilde{P};\widetilde{u}), \end{cases} \tag{3.10}$$

where we set

$$\Gamma(\widetilde{P};\widetilde{u}) = \{\widetilde{v} \in \widetilde{P} \mid \langle \widetilde{u}, \widetilde{v} \rangle = \min_{\widetilde{w} \in \widetilde{P}} \langle \widetilde{u}, \widetilde{w} \rangle\} \subset \widetilde{P}. \tag{3.11}$$

Definition 3.4. *We say that the Laurent polynomial $p(x,s)$ is non-degenerate if for any non-zero $\widetilde{u} \in \mathbb{Z}^n \subset \mathbb{R}^n_{\widetilde{u}}$ the complex hypersurface*

$$\{(x,s) \in T \mid p^{\widetilde{u}}(x,s) = 0\} \tag{3.12}$$

in $T \simeq (\mathbb{C}^*)^n$ is smooth and reduced.

Since $p(x,s)$ is non-degenerate for generic $(c_1, c_2, \dots, c_{j_0-1}, c_{j_0+1}, \dots, c_m) \in \mathbb{C}^{m-1}$, for the calculation of $\zeta_{h,\infty}(j_! R\pi'_! \mathcal{F}[n-1])(t)$ we may assume that $p(x,s)$ is non-degenerate. Now we introduce an equivalence relation $\sim$ of the dual vector space $\mathbb{R}^n_{\widetilde{u}}$ of $\mathbb{R}^n_{\widetilde{v}}$ defined by

$$\widetilde{u} \sim \widetilde{u}' \quad \Longleftrightarrow \quad \Gamma(\widetilde{P};\widetilde{u}) = \Gamma(\widetilde{P};\widetilde{u}'). \tag{3.13}$$

Then we obtain a decomposition $\mathbb{R}^n_{\widetilde{u}} = \bigsqcup \tau$ into equivalence classes τ and a subdivision $\Sigma_1 = \{\overline{\tau}\}$ of $\mathbb{R}^n_{\widetilde{u}}$ into convex cones $\overline{\tau}$. Namely Σ_1 is the dual subdivision of $\widetilde{P} \subset \mathbb{R}^n_{\widetilde{v}}$. Note that Σ_1 is not necessarily a fan in $\mathbb{R}^n_{\widetilde{u}}$ since we do not assume $\dim \widetilde{P} = n$. Next for $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n = \pm 1$ we set

$$\sigma_{\varepsilon_1,\dots,\varepsilon_n} = \{\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2, \dots, \widetilde{u}_n) \mid \varepsilon_i \widetilde{u}_i \geq 0 \text{ for } i = 1, 2, \dots, n\} \tag{3.14}$$

and consider the complete fan Σ_2 in $\mathbb{R}^n_{\widetilde{u}}$ consisting of the cones $\sigma_{\varepsilon_1,\dots,\varepsilon_n}$ $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n = \pm 1)$ and their faces. Let Σ_0 be the common subdivision of Σ_1 and Σ_2. Applying some more subdivisions to Σ_0 if necessary, we obtain a complete fan Σ in $\mathbb{R}^n_{\widetilde{u}}$ such that the toric variety X_Σ associated to it is smooth and complete (see [3], [6] and [18] etc.). Recall that the algebraic torus T acts on X_Σ and there exists a natural bijection between the set of T-orbits in X_Σ and that of the cones $\sigma \in \Sigma$ in Σ. For a cone $\sigma \in \Sigma$ denote by T_σ the corresponding T-orbit in X_Σ. Then we obtain a decomposition $X_\Sigma = \sqcup_{\sigma \in \Sigma} T_\sigma$ of X_Σ into T-orbits. Let $\iota : T \hookrightarrow X_\Sigma$ be the canonical embedding. We can show that the extension of $s = \pi' : T \longrightarrow \mathbb{C}$ to a meromorphic function on X_Σ has no point of indeterminacy as follows. Indeed, for an n-dimensional cone $\sigma_0 \in \Sigma$ denote by $\mathbb{C}^n(\sigma_0)$ $(\simeq \mathbb{C}^n)$ the smooth toric variety associated to the fan consisting of σ_0 and its faces. Then $\mathbb{C}^n(\sigma_0)$ is an affine open subset of X_Σ containing T and X_Σ is covered by such open subsets.

Lemma 3.5. *For any n-dimensional cone $\sigma_0 \in \Sigma$, the meromorphic extension of $s = \pi' : T \longrightarrow \mathbb{C}$ to $\mathbb{C}^n(\sigma_0)$ has no point of indeterminacy.*

Proof. For an n-dimensional cone $\sigma_0 \in \Sigma$, let $\{\widetilde{u}(\sigma_0)^1, \widetilde{u}(\sigma_0)^2, \dots, \widetilde{u}(\sigma_0)^n\}$ be the 1-skelton of σ_0 (i.e. the set of the primitive vectors on the edges of σ_0). Then on $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}^n_y$ the meromorphic extension g of $s = \pi' : T \longrightarrow \mathbb{C}$ has the form

$$g(y) = y_1^{k_1} y_2^{k_2} \cdots y_n^{k_n}, \tag{3.15}$$

where we set

$$k_i = \langle \widetilde{u}(\sigma_0)^i, (0, \dots, 0, 1) \rangle \in \mathbb{Z} \tag{3.16}$$

for $i = 1, 2, \dots, n$. Since Σ is a subdivision of Σ_2, we have

$$k_i \geq 0 \quad (i = 1, 2, \dots, n) \qquad \text{or} \qquad k_i \leq 0 \quad (i = 1, 2, \dots, n). \tag{3.17}$$

$\square$

By this lemma, there exists a holomorphic map $g : X_\Sigma = \overline{T} \longrightarrow \mathbb{P}^1$ such that $g \circ \iota = j \circ \pi'$. Since g is proper, we thus obtain an isomorphism

$$j_! R\pi'_! \mathcal{F} \simeq Rg_* \iota_! \mathcal{F} \tag{3.18}$$

in $\mathbf{D}^b_c(\mathbb{P}^1)$. Then by Proposition 2.7, for the calculation of $\zeta_{h,\infty}(j_! R\pi'_! \mathcal{F})(t) = \zeta_{h,\infty}(Rg_* \iota_! \mathcal{F})(t) \in \mathbb{C}(t)^*$ it suffices to calculate the monodromy zeta function $\zeta_{h \circ g}(\iota_! \mathcal{F})(t) \in \mathbb{C}(t)^*$ of $\iota_! \mathcal{F} \in \mathbf{D}^b_c(X_\Sigma)$ at each point of $g^{-1}(\infty) \subset X_\Sigma$. We can easily see that $g^{-1}(\infty)$ is a union of T-orbits. Let $\sigma \in \Sigma$ be a d-dimensional cone in Σ such that $T_\sigma \subset g^{-1}(\infty)$. We choose an n-dimensional cone $\sigma_0 \in \Sigma$ such that $\sigma \prec \sigma_0$ and let $\{\widetilde{u}(\sigma_0)^1, \widetilde{u}(\sigma_0)^2, \dots, \widetilde{u}(\sigma_0)^n\}$ be its 1-skelton. Let $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}^n_y \subset X_\Sigma$ be the smooth toric variety defined by σ_0 as above. Without loss of generality we may assume that $\{\widetilde{u}(\sigma_0)^1, \dots, \widetilde{u}(\sigma_0)^d\}$ is the 1-skelton

of σ. Then by the condition $T_\sigma \subset g^{-1}(\infty)$ at least one of the vectors $\widetilde{u}(\sigma_0)^1, \dots, \widetilde{u}(\sigma_0)^d$ is contained in the open half space

$$H_{<0} = \{\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2, \dots, \widetilde{u}_n) \mid \widetilde{u}_n < 0\} \subset \mathbb{R}^n_{\widetilde{u}}. \tag{3.19}$$

Since Σ is a subdivision of Σ_2 we get also

$$\sigma \subset \sigma_0 \subset H_{\leq 0} = \{\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2, \dots, \widetilde{u}_n) \mid \widetilde{u}_n \leq 0\}. \tag{3.20}$$

In the affine open subset $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}^n_y$ of X_Σ the $(n-d)$-dimensional T-orbit T_σ is defined by

$$T_\sigma = \{y \in \mathbb{C}^n(\sigma_0) \mid y_1 = \cdots = y_d = 0 \text{ and } y_{d+1}, \dots, y_n \in \mathbb{C}^*\}. \tag{3.21}$$

Denote the meromorphic extension of the Laurent polynomial $p(x,s)$ to X_Σ simply by p. Then on $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}^n_y$ the meromorphic function p has the form

$$p(y) = y_1^{l_1} y_2^{l_2} \cdots y_n^{l_n} \times p_{\sigma_0}(y), \tag{3.22}$$

where we set

$$l_i = \min_{\widetilde{v} \in \widetilde{P}} \langle \widetilde{u}(\sigma_0)^i, \widetilde{v} \rangle \in \mathbb{Z} \tag{3.23}$$

for $i = 1, 2, \dots, n$ and $p_{\sigma_0}(y)$ is a polynomial on $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}^n_y$. By the non-degeneracy of $p(x,s)$ the complex hypersurface $\{y \in \mathbb{C}^n(\sigma_0) \mid p_{\sigma_0}(y) = 0\}$ intersects T_σ transversally. Let $\iota_{\sigma_0} : T \cap \{p_{\sigma_0} \neq 0\} \hookrightarrow \mathbb{C}^n(\sigma_0) \simeq \mathbb{C}^n_y$ be the inclusion. Then on $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}^n_y$ we have an isomorphism

$$\iota_! \mathcal{F} \simeq (\iota_{\sigma_0})_! \mathcal{L}', \tag{3.24}$$

where $\mathcal{L}'$ is a local system on $T \cap \{p_{\sigma_0} \neq 0\}$. Moreover on $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}^n_y$ the meromorphic function g has the form

$$g(y) = y_1^{l'_1} y_2^{l'_2} \cdots y_n^{l'_n}, \tag{3.25}$$

where we set

$$l'_i = \langle \widetilde{u}(\sigma_0)^i, (0, \dots, 0, 1) \rangle \leq 0 \tag{3.26}$$

for $i = 1, 2, \dots, n$. Hence the function $h \circ g$ has the form

$$(h \circ g)(y) = y_1^{-l'_1} y_2^{-l'_2} \cdots y_n^{-l'_n}, \tag{3.27}$$

on $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}^n_y$. By Proposition 2.9, if $d = \dim\sigma \geq 2$ we have $\zeta_{h \circ g, y}(\iota_! \mathcal{F})(t) = 1$ for any $y \in T_\sigma$. This implies that for the calculation of $\zeta_{h,\infty}(Rg_* \iota_! \mathcal{F})(t) \in \mathbb{C}(t)^*$ it suffices to consider only the cones $\sigma \in \Sigma$ of dimension one such that $T_\sigma \subset g^{-1}(\infty)$. From now on, we assume always $d = \dim\sigma = 1$. Denote the unique primitive vector $\widetilde{u}(\sigma_0)^1 \in \mathbb{Z}^n \setminus \{0\}$ on σ simply by $\widetilde{u}(\sigma)$. Then by the condition $T_\sigma \subset g^{-1}(\infty)$ we have $\widetilde{u}(\sigma) \in H_{<0}$. On $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}^n_y$ we have an isomorphism

$$\iota_! \mathcal{F} \simeq (\iota_{\sigma_0})_! \left\{ \mathbb{C}\ y_1^{\delta} \times (y_2^{\rho_2} \cdots y_n^{\rho_n}) \times p_{\sigma_0}(y)^{\alpha} \right\}, \tag{3.28}$$

where $\rho_2, \dots, \rho_n$ are some complex numbers and we set

$$l = \min_{\widetilde{v} \in \widetilde{P}} \langle \widetilde{u}(\sigma), \widetilde{v} \rangle \in \mathbb{Z} \tag{3.29}$$

and $\delta = \alpha l + \langle(\beta, 0), \widetilde{u}(\sigma)\rangle \in \mathbb{C}$. Moreover the meromorphic function $h \circ g : X_\Sigma \longrightarrow \mathbb{P}^1$ has the form

$$(h \circ g)(y) = y_1^{-l'} \times (y_2^{-l'_2} \cdots y_n^{-l'_n}) \tag{3.30}$$

on $\mathbb{C}^n(\sigma_0)$, where we set

$$l' = \langle \widetilde{u}(\sigma), (0, \dots, 0, 1)\rangle < 0. \tag{3.31}$$

Therefore, by Proposition 2.8 and 2.9 for $y \in T_\sigma \simeq (\mathbb{C}^*)^{n-1}$ we have

$$\zeta_{h\circ g, y}(\iota_! \mathcal{F})(t) = \begin{cases} 1 - \exp(2\pi\sqrt{-1}\delta)t^{-l'} & \text{if } y \notin \{p_{\sigma_0} = 0\}\ , \\ 1 & \text{if } y \in \{p_{\sigma_0} = 0\}\ . \end{cases} \tag{3.32}$$

Note that $\Gamma(\widetilde{P}; \widetilde{u}(\sigma))$ is naturally identified with the Newton polytope of the Laurent polynomial $p_{\sigma_0}|_{T_\sigma}$. Hence, if $\dim\Gamma(\widetilde{P}; \widetilde{u}(\sigma)) < \dim T_\sigma = n-1$ the Euler characteristic $\chi(T_\sigma \setminus \{p_{\sigma_0} = 0\})$ of $T_\sigma \setminus \{p_{\sigma_0} = 0\}$ is zero by Bernstein-Khovanskii-Kushnirenko's theorem (see [10] and [11] etc.). This implies that for the calculation of $\zeta_{h,\infty}(Rg_* \iota_! \mathcal{F})(t) \in \mathbb{C}(t)^*$ it suffices to consider only the 1-dimensional cones $\sigma \in \Sigma$ such that $\widetilde{u}(\sigma) \in \mathbb{Z}^n \cap H_{<0}$ and $\dim\Gamma(\widetilde{P}; \widetilde{u}(\sigma)) = n-1$. In order to list up all such 1-dimensional cones $\sigma \in \Sigma$, for $r = 1, 2, \dots, k$ let $\widetilde{\Delta}_r$ be the convex hull of $\Delta_r \subset \mathbb{R}^{n-1}_v \oplus \{0\}$ and the point $(a(j_0), 1)$ in $\mathbb{R}^n_{\widetilde{v}} = \mathbb{R}^{n-1}_v \oplus \mathbb{R}$. We also denote by $\widetilde{Q}$ the convex hull of $Q \subset \mathbb{R}^{n-1}_v \oplus \{0\}$ and $(a(j_0), 1)$. Then $\widetilde{\Delta}_1, \widetilde{\Delta}_2, \dots, \widetilde{\Delta}_k$ are the facets of the n-dimensional polytope $\widetilde{Q}$ whose inner conormal vectors are contained in the open half space $H_{<0} \subset \mathbb{R}^n_{\widetilde{u}}$. For $r = 1, 2, \dots, k$ let $\widetilde{u}^r \in \mathbb{Z}^n \cap H_{<0}$ be the unique non-zero primitive vector such that

$$\widetilde{\Delta}_r = \{\widetilde{v} \in \widetilde{Q} \mid \langle \widetilde{u}^r, \widetilde{v}\rangle = \min_{\widetilde{w}\in\widetilde{Q}} \langle \widetilde{u}^r, \widetilde{w}\rangle\}. \tag{3.33}$$

Then by showing that the vector $(u^r, -d_r) \in \mathbb{Z}^n$ takes the constant value $h_r \in \mathbb{Z}$ on $\widetilde{\Delta}_r$ we can easily prove the following lemma.

Lemma 3.6. *For $r = 1, 2, \dots, k$ we have $\widetilde{u}^r = (u^r, -d_r) \in \mathbb{Z}^n \cap H_{<0}$ and*

$$\min_{\widetilde{v}\in\widetilde{P}} \langle \widetilde{u}^r, \widetilde{v}\rangle = \min_{\widetilde{v}\in\widetilde{Q}} \langle \widetilde{u}^r, \widetilde{v}\rangle = h_r. \tag{3.34}$$

Since $d_r > 0$ is the lattice distance of the point $(a(j_0), 0) \in \widetilde{Q}$ from $\widetilde{\Delta}_r$ by this lemma, we can easily see that the normalized $(n-1)$-dimensional volume $\mathrm{Vol}_{\mathbb{Z}}(\widetilde{\Delta}_r)$ of $\widetilde{\Delta}_r$ is equal to $\mathrm{Vol}_{\mathbb{Z}}(\Delta_r)$. By the definition of $\widetilde{P}$ and $\widetilde{u}^1, \widetilde{u}^2, \dots, \widetilde{u}^k$, for the the calculation of $\zeta_{h,\infty}(Rg_* \iota_! \mathcal{F})(t) \in \mathbb{C}(t)^*$ it suffices to consider only the 1-dimensional cones $\sigma_r = \mathbb{R}_{\geq 0}\widetilde{u}^r$ $(r = 1, 2, \dots, k)$ in Σ. Hence let us consider the case where $\sigma = \sigma_r$ for some $r = 1, 2, \dots, k$ and σ_0 is an n-dimensional cone in Σ such that $\sigma \prec \sigma_0$. Then by Lemma 3.6 on $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}^n_y$ we have an isomorphism

$$\iota_! \mathcal{F} \simeq (\iota_{\sigma_0})_! \left\{ \mathbb{C}\ y_1^{\delta_r} \times (y_2^{\rho_2} \cdots y_n^{\rho_n}) \times p_{\sigma_0}(y)^\alpha \right\}, \tag{3.35}$$

where $\rho_2, \dots, \rho_n$ are some complex numbers and we set $\delta_r = \alpha h_r + \langle \beta, u^r\rangle = \alpha h_r + \langle(\beta, 0), \widetilde{u}^r\rangle \in \mathbb{C}$ as in the introduction. By the same lemma, the function $h \circ g$ has

the form $y_1^{-\langle \widetilde{u}^r,(0,\dots,0,1)\rangle} \times (y_2^{-l'_2} \cdots y_n^{-l'_n}) = y_1^{d_r} \times (y_2^{-l'_2} \cdots y_n^{-l'_n})$ on $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}^n_y$. Then by Proposition 2.8 and 2.9, for $y \in T_\sigma = T_{\sigma_r} \simeq (\mathbb{C}^*)^{n-1}$ we have

$$\zeta_{h\circ g,y}(\iota_! \mathcal{F})(t) = \begin{cases} 1 - \exp(2\pi\sqrt{-1}\delta_r)t^{d_r} & \text{if } y \notin \{p_{\sigma_0} = 0\} \ , \\ 1 & \text{if } y \in \{p_{\sigma_0} = 0\} \ . \end{cases} \tag{3.36}$$

Since the Euler characteristic $\chi(T_{\sigma_r} \setminus \{p_{\sigma_0} = 0\})$ is equal to $(-1)^{n-1}\mathrm{Vol}_{\mathbb{Z}}(\widetilde{\Delta}_r) = (-1)^{n-1}\mathrm{Vol}_{\mathbb{Z}}(\Delta_r)$ by Bernstein-Khovanskii-Kushnirenko's theorem, we obtain the desired result

$$\zeta_{h,\infty}(j_! R\pi_! \mathcal{L}[n-1])(t) = \zeta_{h,\infty}(Rg_* \iota_! \mathcal{F}[n-1])(t) \tag{3.37}$$

$$= \prod_{r=1}^{k} \left\{ 1 - \exp(2\pi\sqrt{-1}\delta_r)t^{d_r} \right\}^{\mathrm{Vol}_{\mathbb{Z}}(\Delta_r)}. \tag{3.38}$$

This completes the proof. $\square$

Example 3.7. *([19, page 25-26]) For the 3×4 integer matrix*

$$M = (m_{ij}) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in M(3, 4, \mathbb{Z}) \tag{3.39}$$

and the vector $\rho = {}^t(\rho_1, \rho_2, \rho_3) = {}^t(c-1, -a, -b) \in \mathbb{C}^3$ consider the following system of partial differential equations on $\mathbb{C}^4_z$.

$$\left(\sum_{j=1}^{4} m_{ij} z_j \frac{\partial}{\partial z_j} - \rho_i \right) f(z) = 0 \quad (1 \leq i \leq 3), \tag{3.40}$$

$$\left\{ \prod_{\mu_j > 0} \left(\frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left(\frac{\partial}{\partial z_j} \right)^{-\mu_j} \right\} f(z) = 0 \quad (\mu \in \mathrm{Ker} M \cap \mathbb{Z}^4 \setminus \{0\}). \tag{3.41}$$

By using the unimodular matrix

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \in SL(3, \mathbb{Z}) \tag{3.42}$$

let us set

$$\widetilde{A} = (a_{ij}) = BM = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in M(3, 4, \mathbb{Z}) \tag{3.43}$$

and $\gamma = {}^t(\gamma_1, \gamma_2, \gamma_3) = B\rho = {}^t(c-1, -a, c-a-b-1) \in \mathbb{C}^3$. Then we obtain an equivalent system

$$\left(\sum_{j=1}^{4} a_{ij} z_j \frac{\partial}{\partial z_j} - \gamma_i \right) f(z) = 0 \quad (1 \leq i \leq 3), \tag{3.44}$$

$$\left\{ \prod_{\mu_j > 0} \left(\frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left(\frac{\partial}{\partial z_j} \right)^{-\mu_j} \right\} f(z) = 0 \quad (\mu \in \mathrm{Ker} \widetilde{A} \cap \mathbb{Z}^4 \setminus \{0\}). \tag{3.45}$$

on $\mathbb{C}_z^4$*. Since the last row of the matrix* $\widetilde{A}$ *is* $(1,1,1,1)$*, this is the* A*-hypergeometric holonomic system* $\mathcal{M}_{A,\gamma}$ *associated to*

$$A=\{(1,0),(0,1),(0,0),(-1,1)\}\subset\mathbb{Z}^2 \tag{3.46}$$

and $\gamma\in\mathbb{C}^3$ *(see the introduction). By Theorem 1.1 for* $j_0=1$*, the characteristic polynomial* $\lambda_1^\infty(t)$ *of the* 1*-st monodromy at infinity of the* A*-hypergeometric functions* $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$ *is given by*

$$\lambda_1^\infty(t)=\left\{t-\exp(2\pi\sqrt{-1}(c-a))\right\}\cdot\left\{t-\exp(2\pi\sqrt{-1}(c-b))\right\}. \tag{3.47}$$

On the other hand, according to [19, page 25-26] the holomorphic solutions $f(z)$ *to* $\mathcal{M}_{A,\gamma}$ *have the form*

$$f(z)=z_1^{c-1}z_2^{-a}z_3^{-b}g\left(\frac{z_1z_4}{z_2z_3}\right), \tag{3.48}$$

where $g(x)$ *satisfy the Gauss hypergeometric equation*

$$x(1-x)\frac{d^2g}{dx^2}(x)+\{c-(a+b+1)x\}\frac{dg}{dx}(x)-abg(x)=0. \tag{3.49}$$

Since the characteristic exponents of this equation at $\infty\in\mathbb{P}$ *are* $a,b\in\mathbb{C}$*, in this very special case we can check that the monodromy at infinity of the restriction of* $\mathrm{Sol}(\mathcal{M}_{A,\gamma})$ *to a generic complex line* $L\simeq\mathbb{C}\subset\mathbb{C}_z^4$ *of the form*

$$L=\{z\in\mathbb{C}^4\mid z_2=c_2,z_3=c_3,z_4=c_4\} \tag{3.50}$$

is given by the formula (3.47) *.*

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